

A new approach to the epsilon expansion of generalized hypergeometric functions

David Greynat ^{a,b} and Javier Sesma ^a

^a*Departamento de Física Teórica, Facultad de Ciencias, Universidad de Zaragoza,
50009 Zaragoza, Spain*

^b*LAPTh., Univ. de Savoie, CNRS, B.P.110, Annecy-le-Vieux F-74941, France*

Abstract

Assumed that the parameters of a generalized hypergeometric function depend linearly on a small variable ε , the successive derivatives of the function with respect to that small variable are evaluated at $\varepsilon = 0$ to obtain the coefficients of the ε -expansion of the function. The procedure, quite naive, benefits from simple explicit expressions of the derivatives, to any order, of the Pochhammer and reciprocal Pochhammer symbols with respect to their argument. The algorithm may be used algebraically, irrespective of the values of the parameters. It reproduces the exact results obtained by other authors in cases of especially simple parameters. Implemented numerically, the procedure improves considerably the numerical expansions given by other methods.

Key words: epsilon expansion, hypergeometric functions, Appell functions, Kampé de Fériet functions, derivatives of Pochhammer symbols.

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<i>Corresponding author:</i>	David Greynat
<i>address:</i>	LAPTh. Univ. de Savoie, CNRS, B.P.110, Annecy-le-Vieux F-74941, France
<i>phone:</i>	33 450 091 692
<i>fax:</i>	33 450 098 913
<i>e-mail:</i>	david.greynat@gmail.com

1 Introduction

Special functions of the hypergeometric class have deserved a considerable interest in the last decade due to their connection with Feynman integrals in quantum field theory. Recurrence relations among hypergeometric functions allow one to reduce the number of Feynman integrals to be computed in a given process. In reciprocity, calculations with Feynman integrals have revealed new relations between hypergeometric functions [1].

Dimensional regularization of Feynman integrals can be conveniently carried out if one knows the ε -expansion of the related generalized hypergeometric functions, that is, an expansion in powers of a small variable ε on which the parameters of the function depend linearly. Several procedures have been suggested to deal with this issue. Algebraic methods applicable in the cases of integer, half integer or rational parameters [2,3,4,5,6,7,8,9] lead to expansions whose coefficients can be written in terms of generalized polylogarithms. Computer packages implementing algorithms based on the Hopf algebra of nested sums [10,11] are available. Let us mention, for instance, a C++ program [12] in the framework of GiNaC, XSUMMER [13], which uses the computer algebra system FORM, HypExp and HypExp 2 [14,15], based on Mathematica, and HYPERDIRE [16], useful for differential reduction of hypergeometric functions. A recently proposed package, NumExp [18], with a different strategy, allows one to get the ε -expansion as a Laurent series whose coefficients are evaluated numerically by using a multi-precision finite difference method.

Here we are concerned with the ε -expansion, to all orders, of generalized hypergeometric functions of one variable, ${}_pF_q$, or of several variables, like the Appell and the Kampé de Fériet functions [17], with parameters depending linearly on ε . No restriction is placed on the parameters, that may be complex. Our procedure stems from the well known series expansions of those functions and, similarly to the method in Ref. [18], consists in obtaining the coefficients of the Laurent series by multiple derivation with respect to ε followed of particularization for $\varepsilon = 0$. Instead of using (approximate) numerical derivation, like in Ref. [18], we benefit from simple formulae for the derivatives, of any order, of the Pochhammer and reciprocal Pochhammer symbols, which allow us to obtain explicit algebraic expressions for the successive terms of the ε -expansion. The validity of our expressions is subordinate to the convergence of the series expansions of the functions considered. We assume in what follows that the values of the variables are such that the needed convergence is guaranteed. Otherwise, such expressions have a purely formal character. Needless to say, analytic continuation allows to cover a wider range of values of the variables.

We present in Section 2 our algorithm. For the sake of clarity, we consider the

ε -expansion of

$${}_pF_q \left(\alpha_1, \dots, \alpha_p \middle| \beta_1, \dots, \beta_q \middle| z \right) \quad (1)$$

with upper and lower parameters of the form

$$\alpha_i = A_i + a_i \varepsilon, \quad i = 1, \dots, p, \quad \beta_j = B_j + b_j \varepsilon, \quad j = 1, \dots, q, \quad (2)$$

without any restriction on the values of A_i , a_i , B_j , b_j . The procedure can be trivially extended to Appell and Kampé de Fériet functions. By way of illustration, we report in Section 3 the results of applying our method to several examples of generalized hypergeometric functions found in the literature. Section 4 contains some pertinent comments. We have devoted Appendices A and B to provide with expressions for the derivatives of the Pochhammer and reciprocal Pochhammer symbols.

2 The algorithm

As said before, our starting point is the series expansion

$${}_pF_q \left(\alpha_1, \dots, \alpha_p \middle| \beta_1, \dots, \beta_q \middle| z \right) = \sum_{m=0}^{\infty} \frac{(\alpha_1)_m \cdots (\alpha_p)_m}{(\beta_1)_m \cdots (\beta_q)_m} \frac{z^m}{m!}, \quad (3)$$

where one uses the Pochhammer symbols, $(\alpha)_n \doteq \Gamma(\alpha + n)/\Gamma(\alpha)$.

We say that all lower parameters β_j are *regular* if all B_j ($j = 1, \dots, q$) are different from zero or a negative integer. In this case the ε -expansion is a Taylor one, free of negative powers of ε . If one or several lower parameters are *singular*, that is, if some of the B_j are zero or a negative integer, negative powers of ε may be present and the expansion is a Laurent one. We treat in turn the two cases.

2.1 All lower parameters are regular

Let us denote by $\mathcal{C}_n(z)$ the coefficients of the ε -expansion

$${}_pF_q \left(\alpha_1, \dots, \alpha_p \middle| z \right) = \sum_{n=0}^{\infty} \mathcal{C}_n(z) \varepsilon^n. \quad (4)$$

Obviously,

$$\mathcal{C}_n(z) = \frac{1}{n!} \frac{\partial^n}{\partial \varepsilon^n} {}_pF_q \left(\alpha_1, \dots, \alpha_p \middle| z \right) \bigg|_{\varepsilon=0}. \quad (5)$$

Let us introduce the abbreviations

$$\mathcal{P}_m^{(k)}(\alpha) \equiv \frac{1}{k!} \frac{d^k}{d\alpha^k}(\alpha)_m, \quad \mathcal{Q}_m^{(k)}(\beta) \equiv \frac{1}{k!} \frac{d^k}{d\beta^k} \frac{1}{(\beta)_m}, \quad (6)$$

to represent the derivatives of the Pochhammer and the reciprocal Pochhammer symbols with respect to their arguments. Then, derivation with respect to ε in Eq. (3) gives

$$\begin{aligned} \frac{\partial^n}{\partial \varepsilon^n} {}_pF_q \left(\alpha_1, \dots, \alpha_p \middle| z \right) &= n! \sum_{m=0}^{\infty} \frac{z^m}{m!} \sum_{k_1=0}^n a_1^{k_1} \mathcal{P}_m^{(k_1)}(\alpha_1) \\ &\quad \times \sum_{k_2=0}^{n_1} a_2^{k_2} \mathcal{P}_m^{(k_2)}(\alpha_2) \cdots \sum_{k_p=0}^{n_{p-1}} a_p^{k_p} \mathcal{P}_m^{(k_p)}(\alpha_p) \\ &\quad \times \sum_{k_{p+1}=0}^{n_p} b_1^{k_{p+1}} \mathcal{Q}_m^{(k_{p+1})}(\beta_1) \sum_{k_{p+2}=0}^{n_{p+1}} b_2^{k_{p+2}} \mathcal{Q}_m^{(k_{p+2})}(\beta_2) \cdots \\ &\quad \cdots \sum_{k_{p+q-1}=0}^{n_{p+q-2}} b_{q-1}^{k_{p+q-1}} \mathcal{Q}_m^{(k_{p+q-1})}(\beta_{q-1}) b_q^{n_{p+q-1}} \mathcal{Q}_m^{(n_{p+q-1})}(\beta_q), \end{aligned} \quad (7)$$

where we have denoted

$$n_l = n - k_1 - k_2 - \dots - k_l, \quad (l = 1, 2, \dots, p+q-1). \quad (8)$$

The expression of $\mathcal{C}_n(z)$ then results immediately from the right hand side of Eq. (7), by suppressing the initial factor $n!$ and substituting A_i for α_i ($i = 1, 2, \dots, p$) and B_j for β_j ($j = 1, 2, \dots, q$). Simple explicit expressions

for $\mathcal{P}_m^{(k)}(A_i)$ and $\mathcal{Q}_m^{(k)}(B_j)$ are obtained in Appendices A and B, respectively. They are

$$\mathcal{P}_0^{(k)}(A_i) = \delta_{k,0}, \quad \mathcal{P}_m^{(k)}(A_i) = 0 \quad \text{for } k > m > 0, \quad (9)$$

$$\mathcal{P}_m^{(k)}(A_i) = (-1)^{m-k} \sum_{l=0}^{m-k} (-1)^l \binom{m}{l} s(m-l, k) (A_i)_l \quad \text{for } m \geq k, \quad (10)$$

$$\mathcal{Q}_0^{(k)}(B_j) = \delta_{k,0}, \quad \mathcal{Q}_m^{(0)}(B_j) = 1/(B_j)_m, \quad (11)$$

$$\mathcal{Q}_m^{(k)}(B_j) = (-1)^k \sum_{l=0}^{m-1} \frac{(-1)^l}{l! (m-1-l)!} \frac{1}{(B_j+l)^{k+1}} \quad \text{for } m, k > 0, \quad (12)$$

The $s(n, k)$ in Eq. (10) represent the Stirling numbers of the first kind [19, §26.8], whose generating relation is

$$[\ln(1+x)]^k = k! \sum_{n=k}^{\infty} s(n, k) \frac{x^n}{n!}, \quad |x| < 1.$$

2.2 Some lower parameters are singular

Let us consider now the case of β_1, \dots, β_r ($0 < r \leq q$) being singular because, for $j = 1, \dots, r$,

$$B_j = -N_j, \quad N_1 \leq \dots \leq N_r \quad \text{nonnegative integers.} \quad (13)$$

The series expansion in Eq. (3) can be written in the form

$$\begin{aligned} {}_pF_q \left(\alpha_1, \dots, \alpha_p \middle| \beta_1, \dots, \beta_q \right| z \Big) &= \sum_{m=0}^{N_1} \frac{(\alpha_1)_m \cdots (\alpha_p)_m}{(\beta_1)_m \cdots (\beta_q)_m} \frac{z^m}{m!} \\ &+ \frac{\varepsilon^{-1}}{b_1} \sum_{m=N_1+1}^{N_2} \frac{(\alpha_1)_m \cdots (\alpha_p)_m}{(\widehat{\beta_1})_m (\beta_2)_m \cdots (\beta_q)_m} \frac{z^m}{m!} \\ &+ \frac{\varepsilon^{-2}}{b_1 b_2} \sum_{m=N_2+1}^{N_3} \frac{(\alpha_1)_m \cdots (\alpha_p)_m}{(\widehat{\beta_1})_m (\widehat{\beta_2})_m (\beta_3)_m \cdots (\beta_q)_m} \frac{z^m}{m!} + \dots \\ &\dots + \frac{\varepsilon^{-r}}{b_1 b_2 \cdots b_r} \sum_{m=N_r+1}^{\infty} \frac{(\alpha_1)_m \cdots (\alpha_p)_m}{(\widehat{\beta_1})_m \cdots (\widehat{\beta_r})_m (\beta_{r+1})_m \cdots (\beta_q)_m} \frac{z^m}{m!}, \quad (14) \end{aligned}$$

where a sum is void if the lower limit of the summation index is larger than the upper one. We have represented by $1/(\widehat{\beta_j})_m$ the “regularized” reciprocal

Pochhammer symbol

$$\frac{1}{(\beta_j)_m} \equiv \frac{b_j \varepsilon}{(\beta_j)_m}, \quad \beta_j = -N_j + b_j \varepsilon, \quad 0 \leq N_j < m, \quad (15)$$

whose derivatives with respect to its variable,

$$\widehat{\mathcal{Q}}_m^{(k)}(\beta_j) \equiv \frac{1}{k!} \frac{d^k}{d\beta_j^k} \frac{1}{(\beta_j)_m}, \quad (16)$$

are calculated in Appendix B. The result is

$$\widehat{\mathcal{Q}}_m^{(k)}(\beta_j) = (-1)^k \sum_{l=0, l \neq N_j}^{m-1} \frac{(-1)^l}{l! (m-1-l)!} \frac{N_j - l}{(\beta_j + l)^{k+1}}, \quad (17)$$

and, for $\varepsilon = 0$,

$$\widehat{\mathcal{Q}}_m^{(k)}(\beta_j) \Big|_{\varepsilon=0} = \widehat{\mathcal{Q}}_m^{(k)}(-N_j) = - \sum_{l=0, l \neq N_j}^{m-1} \frac{(-1)^l}{l! (m-1-l)!} \frac{1}{(N_j - l)^k}, \quad (18)$$

The ε -expansion is now of the form

$${}_pF_q \left(\alpha_1, \dots, \alpha_p \middle| z \right) = \sum_{n=-r}^{\infty} \mathcal{C}_n(z) \varepsilon^n. \quad (19)$$

The coefficients $\mathcal{C}_n(z)$ are immediately obtained as a sum of the corresponding coefficients of the ε -expansions of the sums in the right hand side of Eq. (14), which can be evaluated by means of the algorithm used to obtain Eq. (7) from Eq. (3).

3 Some examples

The procedure sketched above may be used to get algebraic expansions, when the parameters A_i and B_j take especially simple values. Nevertheless, we find our method most suited to obtain numerical expansions, for arbitrary values of the parameters, by means of an implementation of the procedure in FORTRAN or **Mathematica** or any similar language. The algorithm is so simple that every user can construct an efficient package. In this section we consider

several examples for which results, obtained by different methods, are available in the literature. Our procedure is able to reproduce the exact algebraic expansions and improves considerably the numerical ones.

3.1 Algebraic expansion of a Gauss hypergeometric function

As an example of the algebraic use of our algorithm, let us try to obtain the first terms of the ε -expansion of the Gauss hypergeometric function

$${}_2F_1 \left(\begin{matrix} a_1 \varepsilon, a_2 \varepsilon \\ 1 + b_1 \varepsilon \end{matrix} \middle| z \right). \quad (20)$$

According to Eq. (7) and its form for $\varepsilon = 0$, the coefficients of the ε -expansion are given by

$$\begin{aligned} \mathcal{C}_n(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!} \sum_{k_1=0}^n a_1^{k_1} \mathcal{P}_m^{(k_1)}(A_1) \sum_{k_2=0}^{n-k_1} a_2^{k_2} \mathcal{P}_m^{(k_2)}(A_2) \\ \times b_1^{n-k_1-k_2} \mathcal{Q}_m^{(n-k_1-k_2)}(B_1). \end{aligned} \quad (21)$$

In the particular case of Eq. (20), one has $A_1 = 0$, $A_2 = 0$, $B_1 = 1$. For these values of the parameters, Eqs. (9) to (12) reduce to

$$\mathcal{P}_0^{(k)}(0) = \delta_{k,0}, \quad \mathcal{P}_m^{(0)}(0) = \delta_{m,0}, \quad \mathcal{P}_m^{(k)}(0) = 0 \quad \text{for } k > m, \quad (22)$$

$$\mathcal{P}_m^{(k)}(0) = (-1)^{m-k} s(m, k) \quad \text{for } m \geq k > 0, \quad (23)$$

$$\mathcal{Q}_0^{(k)}(1) = \delta_{k,0}, \quad \mathcal{Q}_m^{(0)}(1) = 1/m!, \quad \mathcal{Q}_m^{(1)}(1) = -\frac{1}{m!} \sum_{l=1}^m \frac{1}{l} = -\frac{H_m}{m!}, \quad (24)$$

$$\mathcal{Q}_m^{(k)}(1) = (-1)^k \sum_{l=1}^m \frac{(-1)^{l-1}}{l! (m-l)!} \frac{1}{l^k} \quad \text{for } m, k > 0, \quad (25)$$

where one uses the harmonic number H_m ,

$$H_m = \sum_{l=1}^m \frac{1}{l}. \quad (26)$$

Then, the first coefficients of the ε -expansion are

$$\begin{aligned} \mathcal{C}_0(z) &= \sum_{m=0}^{\infty} \frac{z^m}{m!} \left(\mathcal{P}_m^{(0)}(0) \right)^2 \mathcal{Q}_m^{(0)}(1) \\ &= 1, \end{aligned} \quad (27)$$

$$\begin{aligned} \mathcal{C}_1(z) &= \sum_{m=0}^{\infty} \frac{z^m}{m!} \left[b_1 \left(\mathcal{P}_m^{(0)}(0) \right)^2 \mathcal{Q}_m^{(1)}(1) + (a_1 + a_2) \mathcal{P}_m^{(0)}(0) \mathcal{P}_m^{(1)}(0) \mathcal{Q}_m^{(0)}(1) \right] \\ &= 0, \end{aligned} \quad (28)$$

$$\begin{aligned} \mathcal{C}_2(z) &= \sum_{m=0}^{\infty} \frac{z^m}{m!} \left[b_1^2 \left(\mathcal{P}_m^{(0)}(0) \right)^2 \mathcal{Q}_m^{(2)}(1) + b_1(a_1 + a_2) \mathcal{P}_m^{(0)}(0) \mathcal{P}_m^{(1)}(0) \mathcal{Q}_m^{(1)}(1) \right. \\ &\quad \left. + \left(a_1 a_2 \left(\mathcal{P}_m^{(1)}(0) \right)^2 + (a_1^2 + a_2^2) \mathcal{P}_m^{(0)}(0) \mathcal{P}_m^{(2)}(0) \right) \mathcal{Q}_m^{(0)}(1) \right] \\ &= \sum_{m=0}^{\infty} \frac{z^m}{m!} a_1 a_2 (s(m, 1))^2 \frac{1}{m!}, \end{aligned} \quad (29)$$

which, by using the representation of the Stirling number

$$s(m, 1) = (-1)^{m-1} (m-1)!, \quad (30)$$

and the definition of the dilogarithm function, $\text{Li}_2(z)$ [19, 25.12.1], can be written in the form

$$\mathcal{C}_2(z) = a_1 a_2 \text{Li}_2(z). \quad (31)$$

$$\begin{aligned} \mathcal{C}_3(z) &= \sum_{m=0}^{\infty} \frac{z^m}{m!} \left[b_1^3 \left(\mathcal{P}_m^{(0)}(0) \right)^2 \mathcal{Q}_m^{(3)}(1) + b_1^2(a_1 + a_2) \mathcal{P}_m^{(0)}(0) \mathcal{P}_m^{(1)}(0) \mathcal{Q}_m^{(2)}(1) \right. \\ &\quad \left. + b_1 \left(a_1 a_2 \left(\mathcal{P}_m^{(1)}(0) \right)^2 + (a_1^2 + a_2^2) \mathcal{P}_m^{(0)}(0) \mathcal{P}_m^{(2)}(0) \right) \mathcal{Q}_m^{(1)}(1) \right. \\ &\quad \left. + \left((a_1^2 a_2 + a_1 a_2^2) \mathcal{P}_m^{(1)}(0) \mathcal{P}_m^{(2)}(0) + (a_1^3 + a_2^3) \mathcal{P}_m^{(0)}(0) \mathcal{P}_m^{(3)}(0) \right) \mathcal{Q}_m^{(0)}(1) \right] \\ &= \sum_{m=1}^{\infty} \frac{z^m}{m!} b_1 a_1 a_2 \left(\mathcal{P}_m^{(1)}(0) \right)^2 \mathcal{Q}_m^{(1)}(1) \\ &\quad + \sum_{m=2}^{\infty} \frac{z^m}{m!} a_1 a_2 (a_1 + a_2) \mathcal{P}_m^{(1)}(0) \mathcal{P}_m^{(2)}(0) \mathcal{Q}_m^{(0)}(1) \\ &= a_1 a_2 \left[b_1 \sum_{m=1}^{\infty} \frac{z^m}{m!} (s(m, 1))^2 \left(-\frac{H_m}{m!} \right) \right. \\ &\quad \left. - (a_1 + a_2) \sum_{m=2}^{\infty} \frac{z^m}{m!} s(m, 1) s(m, 2) \frac{1}{m!} \right], \end{aligned} \quad (32)$$

which, by using the representations Eq. (30) and

$$s(m, 2) = (-1)^m (m-1)! H_{m-1} \quad (33)$$

of the Stirling numbers, can be written in the form

$$\begin{aligned}
\mathcal{C}_3(z) &= a_1 a_2 \left[-b_1 \sum_{m=1}^{\infty} \frac{z^m}{m^2} H_m + (a_1 + a_2) \sum_{m=2}^{\infty} \frac{z^m}{m^2} H_{m-1} \right] \\
&= a_1 a_2 \left[-b_1 \sum_{m=1}^{\infty} \frac{z^m}{m^2} \frac{1}{m} + (a_1 + a_2 - b_1) \sum_{m=2}^{\infty} \frac{z^m}{m^2} H_{m-1} \right]. \tag{34}
\end{aligned}$$

The last equation can be expressed in terms of the polylogarithm function, $\text{Li}_n(z)$ [19, Eq. 25.12.10], and the Nielsen's generalized polylogarithms, $S_{n,p}(z)$ [20], to give

$$\mathcal{C}_3(z) = a_1 a_2 [-b_1 \text{Li}_3(z) + (a_1 + a_2 - b_1) S_{1,2}(z)]. \tag{35}$$

We see that our method reproduces the results obtained by Kalmykov [5, Eq. 4.7] by help of relations presented in Ref. [21].

3.2 Algebraic expansion of an Appell function

Del Duca *et al.* [22, Eq. (5.19)] have considered the algebraic ε -expansion of the Appell function

$$F_4 \left(\begin{matrix} 1, 1 + \varepsilon \\ 1 + \varepsilon, 1 + \varepsilon \end{matrix} \middle| x_1, x_2 \right) \tag{36}$$

by using an algorithm based on the algebra properties of nested harmonic sums. From the series expansion of the Appell function

$$F_4 \left(\begin{matrix} \alpha_1, \alpha_2 \\ \beta_1, \beta_2 \end{matrix} \middle| x_1, x_2 \right) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{(\alpha_1)_{m_1+m_2} (\alpha_2)_{m_1+m_2}}{(\beta_1)_{m_1} (\beta_2)_{m_2}} \frac{x_1^{m_1}}{m_1!} \frac{x_2^{m_2}}{m_2!}, \tag{37}$$

application of our method would give for the coefficient of ε^n in the ε -expansion the expression

$$\begin{aligned}
\mathcal{C}_n(x_1, x_2) &= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{x_1^{m_1}}{m_1!} \frac{x_2^{m_2}}{m_2!} \sum_{k_1=0}^n a_1^{k_1} \mathcal{P}_{m_1+m_2}^{(k_1)}(A_1) \sum_{k_2=0}^{n-k_1} a_2^{k_2} \mathcal{P}_{m_1+m_2}^{(k_2)}(A_2) \\
&\quad \times \sum_{k_3=0}^{n-k_1-k_2} b_1^{k_3} \mathcal{Q}_{m_1}^{(k_3)}(B_1) b_2^{n-k_1-k_2-k_3} \mathcal{Q}_{m_2}^{(n-k_1-k_2-k_3)}(B_2), \tag{38}
\end{aligned}$$

which, for the values of the parameters in Eq. (36) reduces to

$$\begin{aligned} \mathcal{C}_n(x_1, x_2) &= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{x_1^{m_1}}{m_1!} \frac{x_2^{m_2}}{m_2!} \mathcal{P}_{m_1+m_2}^{(0)}(1) \sum_{k_2=0}^n \mathcal{P}_{m_1+m_2}^{(k_2)}(1) \\ &\quad \times \sum_{k_3=0}^{n-k_2} \mathcal{Q}_{m_1}^{(k_3)}(1) \mathcal{Q}_{m_2}^{(n-k_2-k_3)}(1), \end{aligned} \quad (39)$$

By using Eqs. (9) and (10) with $A_i = 1$ and Eqs. (24) and (25), we obtain for the first coefficients of the ε -expansion (understanding that a sum is zero if the lower limit of the summation index is larger than the upper one)

$$\begin{aligned} \mathcal{C}_0(x_1, x_2) &= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{x_1^{m_1}}{m_1!} \frac{x_2^{m_2}}{m_2!} \left(\mathcal{P}_{m_1+m_2}^{(0)}(1) \right)^2 \mathcal{Q}_{m_1}^{(0)}(1) \mathcal{Q}_{m_2}^{(0)}(1) \\ &= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \binom{m_1+m_2}{m_1}^2 x_1^{m_1} x_2^{m_2} \\ &= F_4(1, 1; 1, 1; x_1, x_2). \end{aligned} \quad (40)$$

$$\begin{aligned} \mathcal{C}_1(x_1, x_2) &= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{x_1^{m_1}}{m_1!} \frac{x_2^{m_2}}{m_2!} \mathcal{P}_{m_1+m_2}^{(0)}(1) \left[\mathcal{P}_{m_1+m_2}^{(1)}(1) \mathcal{Q}_{m_1}^{(0)}(1) \mathcal{Q}_{m_2}^{(0)}(1) \right. \\ &\quad \left. + \mathcal{P}_{m_1+m_2}^{(0)}(1) \left(\mathcal{Q}_{m_1}^{(1)}(1) \mathcal{Q}_{m_2}^{(0)}(1) + \mathcal{Q}_{m_1}^{(0)}(1) \mathcal{Q}_{m_2}^{(1)}(1) \right) \right] \\ &= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \binom{m_1+m_2}{m_1}^2 x_1^{m_1} x_2^{m_2} [H_{m_1+m_2} - H_{m_1} - H_{m_2}] . \end{aligned} \quad (41)$$

$$\begin{aligned} \mathcal{C}_2(x_1, x_2) &= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{x_1^{m_1}}{m_1!} \frac{x_2^{m_2}}{m_2!} \mathcal{P}_{m_1+m_2}^{(0)}(1) \left[\mathcal{P}_{m_1+m_2}^{(2)}(1) \mathcal{Q}_{m_1}^{(0)}(1) \mathcal{Q}_{m_2}^{(0)}(1) \right. \\ &\quad \left. + \mathcal{P}_{m_1+m_2}^{(1)}(1) \left(\mathcal{Q}_{m_1}^{(1)}(1) \mathcal{Q}_{m_2}^{(0)}(1) + \mathcal{Q}_{m_1}^{(0)}(1) \mathcal{Q}_{m_2}^{(1)}(1) \right) \right. \\ &\quad \left. + \mathcal{P}_{m_1+m_2}^{(0)}(1) \left(\mathcal{Q}_{m_1}^{(2)}(1) \mathcal{Q}_{m_2}^{(0)}(1) + \mathcal{Q}_{m_1}^{(1)}(1) \mathcal{Q}_{m_2}^{(1)}(1) + \mathcal{Q}_{m_1}^{(0)}(1) \mathcal{Q}_{m_2}^{(2)}(1) \right) \right] \\ &= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \binom{m_1+m_2}{m_1}^2 x_1^{m_1} x_2^{m_2} \left[\sum_{l=2}^{m_1+m_2} \frac{H_{l-1}}{l} \right. \\ &\quad \left. - (-1)^{m_1+m_2} H_{m_1+m_2} (H_{m_1} + H_{m_2}) + \left(\sum_{l=1}^{m_1} (-1)^{l-1} \binom{m_1}{l} \frac{1}{l^2} \right) \right. \\ &\quad \left. + H_{m_1} H_{m_2} + \left(\sum_{l=1}^{m_2} (-1)^{l-1} \binom{m_2}{l} \frac{1}{l^2} \right) \right]. \end{aligned} \quad (42)$$

The same coefficients, with a different notation, have been obtained by Del Duca *et al.* [22, Eq. (5.19)]. The reduction of the double sums (in m_1 and

m_2) to known functions does not seem to be an easy task. In [22, Eq. (5.90)] transcendental functions $\mathcal{M}(\vec{i}, \vec{j}, \vec{k}; x_1, x_2)$ have been introduced to represent the different terms of the coefficients of the ε -expansion. The properties of those \mathcal{M} functions are discussed in an Appendix of the same paper.

3.3 Numerical expansion of several generalized hypergeometric functions

As a first example of numerical application of our algorithm, we consider the expansion of

$${}_4F_3 \left(\begin{matrix} -4\varepsilon, -\frac{1}{2} - \varepsilon, -\frac{3}{2} - 2\varepsilon, \frac{1}{2} - 3\varepsilon \\ -\frac{1}{2} + 2\varepsilon, -\frac{1}{2} + 4\varepsilon, \frac{1}{2} + 6\varepsilon \end{matrix} \middle| \frac{1}{2} \right), \quad (43)$$

whose first terms have been obtained by Carter and Heinrich [23] by using the **HypExp** package [14,15] and by Huang and Liu with their own **NumExp** package [18]. Our Table 1 shows their results together with those obtained by means of a pedestrian **Mathematica** implementation of our method. Obviously, the m -sum in the series expansion of the hypergeometric function must be truncated at, say, $m = M$. The value of M determines the accuracy of our results, that may be controlled by successive increments of M . Notice that, as all parameters are rational, **Mathematica** gives, for the truncated m -sum, exact results in the form of rational numbers. The values shown in Table 1 are the decimal approximation of those rational results. For the parameters and variable in Eq. (43) a value of $M = 50$ is enough to give the digits reported.

In the second example,

$${}_5F_4 \left(\begin{matrix} \varepsilon, -\varepsilon, -3\varepsilon, -5\varepsilon, -7\varepsilon \\ 2\varepsilon, 4\varepsilon, 6\varepsilon, 8\varepsilon \end{matrix} \middle| \frac{1}{2} \right), \quad (44)$$

we realize that all lower parameters are singular, as

$$B_1 = B_2 = B_3 = B_4 = 0. \quad (45)$$

Equation (14) adopts then the form

$${}_5F_4 \left(\begin{matrix} \alpha_1, \dots, \alpha_5 \\ \beta_1, \dots, \beta_4 \end{matrix} \middle| z \right) = 1 + \frac{\varepsilon^{-4}}{b_1 b_2 b_3 b_4} \sum_{m=1}^{\infty} \frac{(\alpha_1)_m \cdots (\alpha_5)_m}{(\beta_1)_m \cdots (\beta_4)_m} \frac{z^m}{m!}, \quad (46)$$

Table 1

Coefficients of the ε -expansion of the generalized hypergeometric function given in Eq. (43).

ε order	HypExp	NumExp	our method
ε^0	1	0.9999999999999999	1
ε^1	-4.27969	-4.27968776167885	-4.27968776167886
ε^2	-26.6976	-26.6975474079	-26.6975474079466
ε^3		195.87119	195.871193504205
ε^4		-7313.7	-7313.74176765086
ε^5			90693.2356441548
ε^6			-1426862.01660383
ε^7			17612046.1433229
ε^8			-233969019.148423
ε^9			2846673719.75988
ε^{10}			-35635855655.1898

Nevertheless, this particular case is especially simple, as

$$A_1 = A_2 = A_3 = A_4 = A_5 = 0, \quad (47)$$

and the negative exponent terms of the Laurent expansion become identically equal to zero. Equivalent to the expansion in Eq. (46), we may write the following one

$${}_5F_4 \left(\begin{matrix} \alpha_1, \dots, \alpha_5 \\ \beta_1, \dots, \beta_4 \end{matrix} \middle| z \right) = 1 + \frac{a_1 a_2 a_3 a_4 a_5}{b_1 b_2 b_3 b_4} \varepsilon \sum_{m=1}^{\infty} \frac{(1+\alpha_1)_{m-1} \cdots (1+\alpha_5)_{m-1}}{(1+\beta_1)_{m-1} \cdots (1+\beta_4)_{m-1}} \frac{z^m}{m!}, \quad (48)$$

which can be treated as described in the case of no singular lower parameters. The algebraic procedure gives for the first coefficients of the ε -expansion of

$${}_5F_4 \left(\begin{matrix} a_1 \varepsilon, \dots, a_5 \varepsilon \\ b_1 \varepsilon, \dots, b_4 \varepsilon \end{matrix} \middle| z \right)$$

the expressions

$$\mathcal{C}_0(z) = 1,$$

Table 2

Coefficients of the ε -expansion of the generalized hypergeometric function given in Eq. (44).

ε order	HypExp	NumExp	our method
ε^0	1	0.9999999999999999	1
ε^1	0.189532	0.189532432184360	0.18953243218436
ε^2	-2.299043	-2.2990427423	-2.29904274238202
ε^3	55.46902	55.469019	55.4690190360554
ε^4	-1014.39	-1014.39	-1014.39242265234
ε^5			15729.382951742
ε^6			-216907.17756543
ε^7			2728106.3284185
ε^8			-31818216.953372
ε^9			348410894.51286
ε^{10}			-3617363078.8137

$$\mathcal{C}_1(z) = \frac{a_1 a_2 a_3 a_4 a_5}{b_1 b_2 b_3 b_4} (-\ln(1-z)) ,$$

$$\mathcal{C}_2(z) = \frac{a_1 \cdots a_5}{b_1 \cdots b_4} (a_1 + \dots + a_5 - b_1 - \dots - b_4) \frac{1}{2} [\ln(1-z)]^2 ,$$

and so on. The numerical results of our method, for the values of the parameters and the variable given in Eq. (44), are shown in Table 2, together with those obtained by using HypExp [23] and NumExp [18].

Huang and Liu [18] have proven the power of his method by applying it to the case of irrational parameters. Specifically, they have given the expansion of

$${}_4F_3 \left(\begin{matrix} -4\varepsilon, -\frac{1}{2} - \varepsilon, -\frac{\pi}{2} - 2\varepsilon, \frac{1}{3} - 3\varepsilon \\ -\pi + 2\varepsilon, -\frac{1}{4} + 4\varepsilon, \frac{1}{2} + 6\varepsilon \end{matrix} \middle| \frac{1}{2} \right) . \quad (49)$$

Table 3 shows their results, besides of ours.

4 Final comments

We have presented a direct and intuitive procedure to construct the ε -expansion of any function, of one or several variables, of the hypergeometric class, whenever the function can be written as a convergent power series of the variables.

Table 3

Coefficients of the ε -expansion of the generalized hypergeometric function given in Eq. (49).

ε order	NumExp	our method
ε^0	1.00000000000000000000	1
ε^1	-1.44555526747927	-1.44555526747927
ε^2	3.9383879447	3.938387944727
ε^3	-266.94735	-266.9473544234
ε^4	298.66	298.666582673
ε^5		-56037.4042903
ε^6		-113001.082396

The building blocks are the successive derivatives of Pochhammer and reciprocal Pochhammer symbols particularized for the values of the parameters at $\varepsilon = 0$. We have provided with closed explicit expressions for those derivatives to any order, although recurrence relations, given in Appendices A and B, can be equally used to evaluate them. Each coefficient of the ε -expansion appears as a power series of the variable or variables that can be summed, to give a known function, only in very special cases. This fact, however, is not a serious drawback, in our opinion. In fact, the symbols commonly used to refer to most of the special functions are merely a shorthand to represent a series or an integral. Nevertheless, what makes our procedure to be especially useful is its easy implementation to obtain numerical values of the coefficients. The examples given above show that our method allows one to calculate terms of higher order with a notable accuracy. Besides this, the proposed method is self-contained: it does not need auxiliary procedures like, for instance, differential reduction, even in the case of singular lower parameters.

An important limitation to the use of the algorithm presented in this paper stems from the necessary convergence of the series representation of the function to be expanded in powers of ε . This restricts the applicability of the method to certain domains of the variables. Obviously, one may have recourse to analytic continuation. For instance, in the very common case of ${}_{q+1}F_q$ one could use, for $|z| > 1$, the relation [18, Eq. (10)] [24, Sec. 5.3, Eq. (3)]

$$\begin{aligned}
{}_{q+1}F_q \left(\begin{matrix} \alpha_1, \dots, \alpha_{q+1} \\ \beta_1, \dots, \beta_q \end{matrix} \middle| z \right) &= \frac{\Gamma(\beta_1) \cdots \Gamma(\beta_q)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_{q+1})} \sum_{h=1}^{q+1} \frac{\Gamma(\alpha_h) \prod_{l=1; l \neq h}^{q+1} \Gamma(\alpha_l - \alpha_h)}{\prod_{j=1}^q \Gamma(\beta_j - \alpha_h)} \\
&\times (z^{-1} e^{i\pi})^{\alpha_h} {}_{q+1}F_q \left(\begin{matrix} \alpha_h, \{1 + \alpha_h - \beta_k\}_{k=1, \dots, q} \\ \{1 + \alpha_h - \alpha_k\}_{k=1, \dots, q+1; k \neq h} \end{matrix} \middle| \frac{1}{z} \right). \quad (50)
\end{aligned}$$

The discussion of the analytic continuation of all functions of the hypergeometric class is, however, out of the scope of this paper.

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A Derivatives of the Pochhammer symbol $(\alpha)_m$ with respect to its argument α

We have introduced, in Eq. (6), the notation

$$\mathcal{P}_m^{(k)}(\alpha) \equiv \frac{1}{k!} \frac{d^k}{d\alpha^k} (\alpha)_m, \quad (\text{A.1})$$

As $(\alpha)_m$ is a polynomial of degree m in α ,

$$\mathcal{P}_m^{(k)}(\alpha) = 0 \quad \text{for} \quad k > m. \quad (\text{A.2})$$

We assume in the rest of this appendix that $m \geq k$. A generating function for the $\mathcal{P}_m^{(k)}(\alpha)$ can be immediately obtained from [24, Sec. 6.2.1, Eq. (2)]

$$\sum_{m=0}^{\infty} (\alpha)_m (-t)^m / m! \equiv {}_1F_0(\alpha; ; -t) = (1+t)^{-\alpha}, \quad |t| < 1. \quad (\text{A.3})$$

Derivation, k times, with respect to α gives

$$\sum_{m=0}^{\infty} k! \mathcal{P}_m^{(k)}(\alpha) (-t)^m / m! = (-1)^k (1+t)^{-\alpha} [\ln(1+t)]^k, \quad |t| < 1. \quad (\text{A.4})$$

Then, we have

$$\mathcal{P}_m^{(k)}(\alpha) = \frac{(-1)^{k-m}}{k!} \frac{\partial^m}{\partial t^m} \left((1+t)^{-\alpha} [\ln(1+t)]^k \right) \Big|_{t=0}$$

$$= \frac{(-1)^{k-m}}{k!} \sum_{l=0}^m \binom{m}{l} \left(\frac{\partial^l}{\partial t^l} (1+t)^{-\alpha} \right) \left(\frac{d^{m-l}}{dt^{m-l}} [\ln(1+t)]^k \right) \Big|_{t=0}. \quad (\text{A.5})$$

Obviously,

$$\frac{\partial^l}{\partial t^l} (1+t)^{-\alpha} \Big|_{t=0} = (-1)^l (\alpha)_l. \quad (\text{A.6})$$

On the other hand, as $[\ln(1+t)]^k$ is the generating function of the Stirling numbers [19, Eq. 26.8.8],

$$[\ln(1+t)]^k = k! \sum_{n=k}^{\infty} s(n, k) t^n / n!, \quad |t| < 1, \quad (\text{A.7})$$

we get

$$\left(\frac{d^{m-l}}{dt^{m-l}} [\ln(1+t)]^k \right) \Big|_{t=0} = k! s(m-l, k). \quad (\text{A.8})$$

Substitution of Eqs. (A.6) and (A.8) in (A.5) gives

$$\mathcal{P}_m^{(k)}(\alpha) = (-1)^{m-k} \sum_{l=0}^{m-k} (-1)^l \binom{m}{l} s(m-l, k) (\alpha)_l \quad \text{for } m \geq k. \quad (\text{A.9})$$

For numerical computation, however, it may be preferable to make use of the recurrence relation

$$\mathcal{P}_{m+1}^{(k)}(\alpha) = (\alpha + m) \mathcal{P}_m^{(k)}(\alpha) + \mathcal{P}_m^{(k-1)}(\alpha), \quad k > 0, \quad (\text{A.10})$$

with starting values

$$\mathcal{P}_0^{(k)}(\alpha) = \delta_{k,0}, \quad \mathcal{P}_m^{(0)}(\alpha) = (\alpha)_m. \quad (\text{A.11})$$

B Derivatives of the reciprocal Pochhammer symbol $1/(\beta)_m$ with respect to its argument β

The notation

$$\mathcal{Q}_m^{(k)}(\beta) \equiv \frac{1}{k!} \frac{d^k}{d\beta^k} \frac{1}{(\beta)_m} \quad (\text{B.1})$$

has been proposed in Eq. (6) to represent the derivatives of the reciprocal Pochhammer symbol with respect to its variable. Very simple expressions for these $\mathcal{Q}_m^{(k)}(\beta)$ can be easily obtained from the relation [25, Eq. 4.2.2.45]

$$\frac{1}{(\beta)_m} = \sum_{l=0}^{m-1} \frac{(-1)^l}{l! (m-1-l)!} \frac{1}{\beta+l}, \quad m > 0. \quad (\text{B.2})$$

Direct derivation with respect to β in this equation gives

$$\mathcal{Q}_m^{(k)}(\beta) = (-1)^k \sum_{l=0}^{m-1} \frac{(-1)^l}{l! (m-1-l)!} \frac{1}{(\beta+l)^{k+1}}, \quad m > 0. \quad (\text{B.3})$$

provided β is different from a nonpositive integer, $-N$, such that $0 \leq N < m$. To deal with this case (in the limit $\varepsilon \rightarrow 0$), we have defined, in Eq. (15), the regularized reciprocal Pochhammer symbol

$$\frac{1}{\widehat{(\beta)}_m} \equiv \frac{b\varepsilon}{(\beta)_m}, \quad \beta = -N + b\varepsilon, \quad 0 \leq N < m. \quad (\text{B.4})$$

whose derivatives have been denoted, in Eq. (16), by

$$\widehat{\mathcal{Q}}_m^{(k)}(\beta) \equiv \frac{1}{k!} \frac{d^k}{d\beta^k} \frac{1}{\widehat{(\beta)}_m} \quad (\text{B.5})$$

From Eqs. (B.2) and (B.4) we obtain

$$\frac{1}{\widehat{(\beta)}_m} = \delta_{m,1} + \sum_{l=0}^{m-1} \frac{(-1)^l}{l! (m-1-l)!} \frac{N-l}{\beta+l}. \quad (\text{B.6})$$

Repeated derivation with respect to β leads to

$$\widehat{\mathcal{Q}}_m^{(k)}(\beta) = (-1)^k \sum_{l=0, l \neq N}^{m-1} \frac{(-1)^l}{l! (m-1-l)!} \frac{N-l}{(\beta+l)^{k+1}}, \quad k > 0. \quad (\text{B.7})$$

For numerical implementation, one may use the recurrence relations

$$\mathcal{Q}_{m+1}^{(k)}(\beta) = \left(\mathcal{Q}_m^{(k)}(\beta) - \mathcal{Q}_{m+1}^{(k-1)}(\beta) \right) / (\beta+m), \quad (\text{B.8})$$

with initial values

$$\mathcal{Q}_0^{(k)}(\beta) = \delta_{k,0}, \quad \mathcal{Q}_m^{(0)}(\beta) = 1/(\beta)_m, \quad (\text{B.9})$$

and, for the regularized reciprocal Pochhammer symbols when $\beta = -N + b\varepsilon$,

$$\widehat{\mathcal{Q}}_{m+1}^{(k)}(\beta) = \left(\widehat{\mathcal{Q}}_m^{(k)}(\beta) - \widehat{\mathcal{Q}}_{m+1}^{(k-1)}(\beta) \right) / (\beta + m), \quad m > N, \quad (\text{B.10})$$

starting with

$$\widehat{\mathcal{Q}}_{N+1}^{(k)}(\beta) = \mathcal{Q}_N^{(k)}(\beta), \quad \widehat{\mathcal{Q}}_m^{(0)}(\beta) = 1/\widehat{(\beta)}_m, \quad m > N. \quad (\text{B.11})$$

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